Current Distribution and Random Matrix Ensembles for an Integrable Asymmetric Fragmentation Process

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We calculate the time-evolution of a discrete-time fragmentation process in which clusters of particles break up and reassemble and move stochastically with size-dependent rates. In the continuous-time limit the process turns into the totally asymmetric simple exclusion process (only pieces of size 1 break off a given cluster). We express the exact solution of the master equation for the process in terms of a determinant which can be derived using the Bethe ansatz. From this determinant we compute the distribution of the current across an arbitrary bond which after appropriate scaling is given by the distribution of the largest eigenvalue of the Gaussian unitary ensemble of random matrices. This result confirms universality of the scaling form of the current distribution in the KPZ universality class and suggests that there is a link between integrable particle systems and random matrix ensembles.

1. INTRODUCTION

Asymmetric exclusion processes are paradigmatic models for systems far from equilibrium, both for their wide range of applications and the availability of exact results[1,2]. Despite their greatly reduced complexity they capture various generic nonequilibrium phenomena such as the occurrence of shocks and particle condensation[3–5]. A major breakthrough in the exact calculation of universal properties came with the realization that various ensembles of random matrices occur in the study of current fluctuations and related quantities in the totally asymmetric simple exclusion

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process (TASEP)[6,7] and also in the polynuclear growth model[8–11]. Here we use results from random matrix theory combined with the Bethe ansatz to study a process introduced by Brankov *et al.*[12] which describes the stochastic fragmentation and reassembly of diffusing particle clusters in discrete time. Alternatively it may be regarded as a totally asymmetric exclusion process with ordered sequential update.

In the totally asymmetric fragmentation process[12] one considers a one-dimensional lattice where each lattice point is occupied by at most one particle. A string of n consecutive particles (and bounded by empty sites) is considered a cluster of size n. The stochastic time evolution occurs in discrete time steps. From each cluster a piece of size $n' < n$ may break off and move to the right by one lattice unit with probability $(1-p)p^{n'}$. The whole cluster may hop with probability p^n. According to this definition the transition rules are the following:

$$0\underbrace{AA\ldots AAA}_{n}0 \to 0\underbrace{AA\ldots AA}_{n-1}0A \quad \text{with probability } p(1-p)$$

$$0\underbrace{AA\ldots AAA}_{n}0 \to 0\underbrace{AA\ldots A}_{n-2}0AA \quad \text{with probability } p^2(1-p)$$

$$\vdots$$

$$0\underbrace{AA\ldots AAA}_{n}0 \to 0A0\underbrace{A\ldots AAA}_{n-1} \quad \text{with probability } p^{n-1}(1-p)$$

$$0\underbrace{AA\ldots AAA}_{n}0 \to 00\underbrace{AA\ldots AAA}_{n} \quad \text{with probability } p^n.$$

Notice that two clusters can merge through hopping if they are separated by only one vacant site. Other fragmentation processes have been studied recently[13–16].

In the noiseless limit $p = 1$ neither fragmentation nor recombination occurs and all clusters move ballistically with probability 1. In the limit $p \to 0$ with an appropriate rescaling of time ($t \to \infty$ with $t' = pt$ fixed) only pieces of size 1, i.e., single particles may break off. This process is equivalent to the usual TASEP. In discrete time the fragmentation process may be interpreted as an asymmetric exclusion process with long-range hopping. This is forbidden in the usual discrete-time TASEP with parallel update studied in ref. 6. Moreover, the fragmentation process has no particle-hole symmetry and the stationary distribution is a product measure with constant density ρ and stationary current[12]

$$j = p\frac{\rho(1-\rho)}{1-p\rho}. \tag{1}$$

The absence of correlations implies that fragmentation and recombination balance each other such that no condensation into macroscopic clusters occurs. Indeed, our interest is not in the stationary state of the system but how it evolves from a fully "phase-separated" initial state where the left half of the infinite lattice (all sites $k \leqslant 0$) is occupied while the right half (all sites $k > 0$) is vacant.

Under Eulerian scaling (lattice constant a and time step τ tend to zero with the ratio a/τ kept constant) one expects the coarse-grained density $\rho(x, t)$ to be governed by the hydrodynamic conservation law

$$\partial_t \rho + \partial_x j(\rho) = 0 \qquad (2)$$

with the macroscopic current (1). Our interest is in the microscopic fluctuations of the current which after appropriate rescaling is expected to be given by an universal scaling function. In particular, one obtains universal corrections to (2) below the Euler scale.

In Section 2 we first consider the continuous-time limit of the model, i.e., the TASEP. We present a new derivation of the result Eq. (1.18) of ref. 6 for the exact distribution of the time-integrated current. We do not use the combinatorial arguments employed by Johansson, but show how the same expression can be obtained logically independently from the Bethe ansatz solution via the determinant expression of ref. 17. In Section 3 we extend this approach to calculate the current distribution for the fragmentation process with arbitrary fragmentation parameter p. In Section 4 we present some conclusions. Some properties of the functions we use in the calculation are given in the appendices.

2. CONTINUOUS TIME TASEP

2.1. Known Results

2.1.1. Current Fluctuations

Consider the TASEP on an infinite chain in continuous time where particles hop to the right with rate 1, provided the target site is vacant. At $t = 0$ the left half of the system (from site $-\infty$ to site 0) is occupied while the right half is empty. We focus on the probability $P(M, N, t)$ that the Nth particle (which was on the $(1 - N)$th site of the infinite cluster at $t = 0$) hops at least M times up to time t. Using combinatorial arguments involving the longest increasing subsequences of random permutations Johansson has proved (Eq. (1.18) in ref. 6)

$$P(M, N, t) = \frac{1}{Z'_{M,N}} \int_{[0,t]^N} \prod_{1 \leqslant i < j \leqslant N} (x_i - x_j)^2 \prod_{j=1}^{N} x_j^{M-N} e^{-x_j} d^N x \qquad (3)$$

for $M \geqslant N$. To have proper normalization the partition sum has to be

$$Z'_{M,N} = \int_{[0,\infty]^N} \prod_{1 \leqslant i < j \leqslant N} (x_i - x_j)^2 \prod_{j=1}^{N} x_j^{M-N} e^{-x_j} d^N x. \qquad (4)$$

The expression (3) is equal to the probability that the largest eigenvalue of a random matrix AA^* is $\leqslant t$ where A is a $N \times M$ matrix of complex Gaussian random variables with mean zero and variance $1/2$[18].

Let $J(x, t)$ be the number of particles that have crossed the lattice bond $(x, x+1)$ up to time t, i.e., the time-integrated current. By construction one has for the probability that $J(x, t) > m$

$$\text{Prob}[J(x, t) > m] = P(m + x + 1, m + 1, t) \quad \text{for } x \geqslant 0. \qquad (5)$$

2.1.2. Bethe ansatz

Consider the TASEP on the infinite chain with a finite (N) number of particles located initially at sites $A_N = \{l_1, l_2, \ldots, l_N\}$ $(l_1 < l_2 < \cdots < l_N)$. It is proved in ref. 17 that the probability of having these particles on sites $B_N = \{k_1, k_2, \ldots, k_N\}$ $(k_1 < k_2 < \cdots < k_N)$ at time t is given by the determinant

$$Q(A_N, B_N; t) = \begin{vmatrix} F_0(k_1 - l_1; t) & F_{-1}(k_1 - l_2; t) & \cdots & F_{-N+1}(k_1 - l_N; t) \\ F_1(k_2 - l_1; t) & F_0(k_2 - l_2; t) & \cdots & F_{-N+2}(k_2 - l_N; t) \\ \vdots & \vdots & & \vdots \\ F_{N-1}(k_N - l_1; t) & F_{N-2}(k_N - l_2; t) & \cdots & F_0(k_N - l_N; t) \end{vmatrix}. \qquad (6)$$

For the definition and some properties of the $F_p(n; t)$ functions see Appendix 4.

The determinant has been obtained from a coordinate Bethe ansatz for the conditional probability $Q(A_N, B_N; t)$. Given the determinant, the proof that it is the solution of the master equation for the TASEP follows from standard relations for determinants, for details see ref. 17. In the next subsection we show how (3) can be derived directly from (6) without reference to combinatorial properties of the process.

2.2. Calculation

The dynamics of the rightmost N particles in the TASEP are inde-pendent of all particles to their left. Therefore $P(M, N, t)$ of Section 2.1.1 can be expressed via $Q(A_N, B_N; t)$ of Section 2.1.2:

$$P(M, N, t) = \sum_{M-N < k_1 < k_2 < \ldots < k_N} Q(\{-N+1, -N+2, \ldots, 0\},$$
$$\{k_1, k_2, \ldots, k_N\}; t). \tag{7}$$

Inserting (3) and (6) one gets

$$\frac{1}{Z'_{M,N}} \int_{[0,t]^N} \prod_{1 \leqslant i < j \leqslant N} (x_i - x_j)^2 \prod_{j=1}^{N} x_j^{M-N} e^{-x_j} d^N x$$

$$= \sum_{M-N < k_1 < k_2 < \cdots < k_N} \begin{vmatrix} F_0(k_1+N-1;t) & F_{-1}(k_1+N-2;t) & \cdots & F_{-N+1}(k_1;t) \\ F_1(k_2+N-1;t) & F_0(k_2+N-2;t) & \cdots & F_{-N+2}(k_2;t) \\ \vdots & \vdots & & \vdots \\ F_{N-1}(k_N+N-1;t) & F_{N-2}(k_N+N-2;t) & \cdots & F_0(k_N;t) \end{vmatrix}. \tag{8}$$

In what follows we show by determinant manipulations and using proper-ties of the functions F_k that this equality holds.

Our starting point is the rhs of (8). The summation over $(k_1, k_2, \ldots, k_N)$ can be done in N steps for which we choose the following sequence:

$$\sum_{M-N < k_1 < k_2 < \cdots < k_N} = \sum_{k_N=M}^{\infty} \sum_{k_{N-1}=M-1}^{k_N-1} \cdots \sum_{k_2=M-N+2}^{k_3-1} \sum_{k_1=M-N+1}^{k_2-1}. \tag{9}$$

After summation over k_1 (for which we use (A.5)) the first row of the matrix becomes

$$F_1(M; t) - F_1(k_2 + N - 1; t) \quad F_0(M-1; t) - F_0(k_2+N-2; t) \cdots$$
$$F_{-N+2}(M-N+1; t) - F_{-N+2}(k_2; t), \tag{10}$$

which reduces to

$$F_1(M; t) \quad F_0(M-1; t) \cdots F_{-N+2}(M-N+1; t) \tag{11}$$

after adding the second row to it. The same method can be used up to the sum over k_{N-1}. For the last sum we use (A.7) and finally we get

$$
\begin{vmatrix}
F_1(M;t) & F_0(M-1;t) & \cdots F_{-N+2}(M-N+1;t) \\
F_2(M+1;t) & F_1(M;t) & \cdots F_{-N+3}(M-N+2;t) \\
\vdots & \vdots & \vdots \\
F_N(M+N-1;t) & F_{N-1}(M+N-2;t) \cdots & F_1(M;t)
\end{vmatrix}
\tag{12}
$$

for the rhs of (8).

It turns out to be useful to represent all the F functions in (12) by an integral using (A.4). Since for $M \geqslant N$ all are zero at $t=0$ the determinant (12) can be written as

$$
\begin{vmatrix}
\int_0^t F_0(M-1;\tau)d\tau & \int_0^t F_{-1}(M-2;\tau)d\tau & \cdots & \int_0^t F_{-N+1}(M-N;\tau)d\tau \\
\int_0^t F_1(M;\tau)d\tau & \int_0^t F_0(M-1;\tau)d\tau & \cdots & \int_0^t F_{-N+2}(M-N+1;\tau)d\tau \\
\vdots & \vdots & & \vdots \\
\int_0^t F_{N-1}(M+N-2;\tau)d\tau & \int_0^t F_{N-2}(M+N-3;\tau)d\tau & \cdots & \int_0^t F_0(M-1;\tau)d\tau
\end{vmatrix}.
\tag{13}
$$

In the second row we perform a partial integration after which the ith element becomes

$$
tF_{2-i}(M-i+1;t) - \int_0^t \tau F_{1-i}(M-i;\tau)d\tau.
\tag{14}
$$

Note that the constant part is t times the corresponding element of the first row so after subtracting t times the first row the second row becomes

$$
-\int_0^t \tau F_0(M-1;\tau)d\tau \quad -\int_0^t \tau F_{-1}(M-2;\tau)d\tau \quad \cdots
$$
$$
-\int_0^t \tau F_{-N+1}(M-N;\tau)d\tau.
\tag{15}
$$

In the third row we perform a double partial integration. Then we add $t^2/2$ times the first row and subtract t times the (original) second row. Repeating the same procedure for all of the rows we get

$$
\begin{vmatrix}
\int_0^t F_0(M-1;\tau)d\tau & \int_0^t F_{-1}(M-2;\tau)d\tau & \cdots & \int_0^t F_{-N+1}(M-N;\tau)d\tau \\
-\int_0^t \tau F_0(M-1;\tau)d\tau & -\int_0^t \tau F_{-1}(M-2;\tau)d\tau & \cdots & -\int_0^t \tau F_{-N+1}(M-N;\tau)d\tau \\
\frac{1}{2}\int_0^t \tau^2 F_0(M-1;\tau)d\tau & \frac{1}{2}\int_0^t \tau^2 F_{-1}(M-2;\tau)d\tau & \cdots & \frac{1}{2}\int_0^t \tau^2 F_{-N+1}(M-N;\tau)d\tau \\
-\frac{1}{3!}\int_0^t \tau^3 F_0(M-1;\tau)d\tau & -\frac{1}{3!}\int_0^t \tau^3 F_{-1}(M-2;\tau)d\tau & \cdots & -\frac{1}{3!}\int_0^t \tau^3 F_{-N+1}(M-N;\tau)d\tau \\
\vdots & \vdots & & \vdots \\
\frac{\pm 1}{(N-1)!}\int_0^t \tau^{N-1} F_0(M-1;\tau)d\tau & \frac{\pm 1}{(N-1)!}\int_0^t \tau^{N-1} F_{-1}(M-2;\tau)d\tau & \cdots & \frac{\pm 1}{(N-1)!}\int_0^t \tau^{N-1} F_{-N+1}(M-N;\tau)d\tau
\end{vmatrix}
$$

$$
= (-1)^{\left[\frac{N}{2}\right]} \prod_{i=1}^{N-1} \frac{1}{i!} \int_{[0,t]^N} \tau_1^0 \tau_2^1 \tau_3^2 \tau_4^3 \cdots \tau_N^{N-1}
\begin{vmatrix}
F_0(M-1;\tau_1) & F_{-1}(M-2;\tau_1) & \cdots & F_{-N+1}(M-N;\tau_1) \\
F_0(M-1;\tau_2) & F_{-1}(M-2;\tau_2) & \cdots & F_{-N+1}(M-N;\tau_2) \\
F_0(M-1;\tau_3) & F_{-1}(M-2;\tau_3) & \cdots & F_{-N+1}(M-N;\tau_3) \\
\vdots & \vdots & & \vdots \\
F_0(M-1;\tau_N) & F_{-1}(M-2;\tau_N) & \cdots & F_{-N+1}(M-N;\tau_N)
\end{vmatrix} d^N\tau.
$$

$$(16)$$

Using (A.6) we can rewrite $F_{-1}(M-2;\tau_i)$ as $F_0(M-2;\tau_i) - F_0(M-1;\tau_i)$. So adding the first column to the second one the ith element of the latter becomes $F_0(M-2;\tau_i)$. Similar transformations can be done with the other columns as well: since according to (A.8)

$$
F_p(n;t) = \sum_{m=0}^{-p} (-1)^m \binom{-p}{m} F_0(n+m;t) \quad \text{for } p \leqslant 0, \tag{17}
$$

one can add suitable linear combinations of the first $l-1$ columns to the lth one so that the ith element of the lth column becomes

$$
F_0(M-l;\tau_i) = e^{-\tau_i} \frac{\tau_i^{M-l}}{(M-l)!}. \tag{18}
$$

After these transformations one gets for (16):

$$
(-1)^{\left[\frac{N}{2}\right]} \prod_{i=1}^{N-1} \frac{1}{i!} \prod_{i=1}^{N} \frac{1}{(M-i)!}
$$

$$
\times \int_{[0,t]^N} \prod_{i=1}^{N} \left(\tau_i^{M-N} e^{-\tau_i} \right) \tau_1^0 \tau_2^1 \tau_3^2 \tau_4^3 \cdots \tau_N^{N-1}
\begin{vmatrix}
\tau_1^{N-1} & \tau_1^{N-2} & \cdots & \tau_1^1 & \tau_1^0 \\
\tau_2^{N-1} & \tau_2^{N-2} & \cdots & \tau_2^1 & \tau_2^0 \\
\vdots & \vdots & & \vdots & \vdots \\
\tau_N^{N-1} & \tau_N^{N-2} & \cdots & \tau_N^1 & \tau_N^0
\end{vmatrix} d^N\tau.
$$

$$(19)$$

Since the integration is symmetric in the τ_i while the determinant is antisymmetric we may replace the product of the τ_i by the antisymmetric combination

$$\frac{1}{N!}\begin{vmatrix} \tau_1^0 & \tau_1^1 & \cdots & \tau_1^{N-2} & \tau_1^{N-1} \\ \tau_2^0 & \tau_2^1 & \cdots & \tau_2^{N-2} & \tau_2^{N-1} \\ \vdots & \vdots & & \vdots & \vdots \\ \tau_N^0 & \tau_N^1 & \cdots & \tau_N^{N-2} & \tau_N^{N-1} \end{vmatrix} = \frac{(-1)^{\left[\frac{N}{2}\right]}}{N!}\begin{vmatrix} \tau_1^{N-1} & \tau_1^{N-2} & \cdots & \tau_1^1 & \tau_1^0 \\ \tau_2^{N-1} & \tau_2^{N-2} & \cdots & \tau_2^1 & \tau_2^0 \\ \vdots & \vdots & & \vdots & \vdots \\ \tau_N^{N-1} & \tau_N^{N-2} & \cdots & \tau_N^1 & \tau_N^0 \end{vmatrix}. \tag{20}$$

This is the Vandermonde determinant

$$\begin{vmatrix} \tau_1^{N-1} & \tau_1^{N-2} & \cdots & \tau_1^1 & \tau_1^0 \\ \tau_2^{N-1} & \tau_2^{N-2} & \cdots & \tau_2^1 & \tau_2^0 \\ \vdots & \vdots & & \vdots & \vdots \\ \tau_N^{N-1} & \tau_N^{N-2} & \cdots & \tau_N^1 & \tau_N^0 \end{vmatrix} = \prod_{1\leqslant i<j\leqslant N}(\tau_i - \tau_j) \tag{21}$$

and leads to

$$P(M,N,t) = \prod_{i=1}^N\left(\frac{1}{i!(M-i)!}\right)\int_{[0,t]^N}\prod_{i=1}^N\left(\tau_i^{M-N}e^{-\tau_i}\right)\prod_{1\leqslant i<j\leqslant N}(\tau_i-\tau_j)^2 d^N\tau. \tag{22}$$

This is in agreement with the lhs of (8), moreover we get the partition function as a "by-product":

$$Z'_{M,N} = \prod_{i=1}^N i!(M-i)! \tag{23}$$

2.3. Asymptotic Form of the Density Profile

For the TASEP the current (1) reduces to $j=\rho(1-\rho)$ and the density profile of the step-function initial state evolves on Euler scale according to

$$\rho(x,t) = \frac{1}{2}(1-v) \tag{24}$$

where $v=x/t$ and $-1\leqslant v\leqslant 1$[19]. Outside this range the density keeps its initial value.

To describe the density profile below Euler scale we use the asymptotic form of the result (5) for the current distribution. It can be shown that that the formula (22) can be written as a Fredholm determinant with the Meixner kernel which in a proper limit reduces to the Airy kernel (for details see ref. 6). This implies that in this limit the asymptotic form of $P(M, N, t)$ is given by the Tracy and Widom distribution[20] of the Gaussian unitary ensemble (F_{GUE}), viz.

$$\lim_{N \to \infty} P([\gamma N], N, \omega(\gamma)N + \sigma(\gamma)N^{1/3}s) = F_{\text{GUE}}(s) \tag{25}$$

with

$$\omega(\gamma) = \left(1 + \sqrt{\gamma}\right)^2 \quad \text{and} \quad \sigma(\gamma) = \gamma^{-1/6}\left(1 + \sqrt{\gamma}\right)^{4/3}. \tag{26}$$

This type of scaling is characteristic for one-dimensional surface-growth models of the KPZ universality class[8].

For our initial state the integrated current $J(x, t)$ gives the number of particles being at sites $k > x$ at time t (for $x \geqslant 0$). It can be shown using Theorem 1.6 of ref. 6 that

$$\lim_{t \to \infty} \text{Prob}\left[J([vt], t) \leqslant \frac{t}{4}(1 - v)^2 + 2^{-4/3}\left(1 - v^2\right)^{2/3}t^{1/3}s\right] = 1 - F_{\text{GUE}}(-s) \tag{27}$$

for $0 \leqslant v < 1$. Because of particle-hole symmetry, one has a similar expression for $-1 < v \leqslant 0$.

We can also consider the corresponding surface growth model where $h(x, t)$ is defined as

$$h(x, t) = |x| + 2J(x, t). \tag{28}$$

The asymptotic mean shape of $h([vt], t)/t = (1 + v^2)/2$ on Euler scale follows directly from (24). The deviations can be calculated from (27):

$$\lim_{t \to \infty} \text{Prob}\left[\frac{t}{2}(1 + v^2) - h([vt], t) < 2^{-1/3}\left(1 - v^2\right)^{2/3}t^{1/3}s\right] = F_{\text{GUE}}(s) \tag{29}$$

for $|v| < 1$. This implies that

$$\frac{\frac{t}{2}(1 + v^2) - \overline{h}([vt], t)}{2^{-1/3}(1 - v^2)^{2/3}t^{1/3}} = -c, \tag{30}$$

where $\bar{h}(x, t)$ is the mean height at position x and time t, and

$$-c = \int_{-\infty}^{\infty} ds\, s\, F'_{\text{GUE}}(s) = -1.77109, \tag{31}$$

is the mean of the distribution F_{GUE}. Since the density $\rho(x, t)$ is $\frac{1}{2}(1 - \bar{h}(x, t) + \bar{h}(x-1, t))$ expression (30) allows us to calculate the correction to the density profile (24)

$$\rho(vt, t) = \frac{1-v}{2} + c\frac{2^{2/3}}{3}v\left(1 - v^2\right)^{-1/3} t^{-2/3} \tag{32}$$

for $|v| < 1$. The KPZ exponent 2/3 appearing here is universal. Note that the coefficient $v\left(1 - v^2\right)^{-1/3}$ diverges as $v \to \pm 1$. In these singular points we expect a $t^{-1/2}$ correction[21].

3. THE DISCRETE-TIME FRAGMENTATION PROCESS

3.1. Complete Solution of the Master-equation by Bethe ansatz

The solution is very similar to the one of the continuous time TASEP. As in ref. 17 one constructs the Bethe solution for the master equation for the conditional probabilities and recasts the expression in terms of a determinant. Then one proves by elementary matrix manipulations that the analogue of Eq. (6) is the following (for details see ref. 22):

$$Q(A_N, B_N; t) = \begin{vmatrix} D_0(k_1 - l_1; t) & D_{-1}(k_1 - l_2; t) & \cdots & D_{-N+1}(k_1 - l_N; t) \\ D_1(k_2 - l_1; t) & D_0(k_2 - l_2; t) & \cdots & D_{-N+2}(k_2 - l_N; t) \\ \vdots & \vdots & & \vdots \\ D_{N-1}(k_N - l_1; t) & D_{N-2}(k_N - l_2; t) & \cdots & D_0(k_N - l_N; t) \end{vmatrix}. \tag{33}$$

For the definition and the main properties of the D functions see Appendix B.

3.2. Calculation of *P(M, N, t)*

We follow the strategy of the previous section, but some care needs to be taken when manipulating sums rather than the time integrals. Here $P(M, N, t)$ is again defined by (7) but now with the Q of (33). The summation gives the same result as for the continuous time version since the

corresponding properties of the D functions are the same as those of the F functions:

$$P(M, N, t) = \begin{vmatrix} D_1(M; t) & D_0(M - 1; t) & \cdots & D_{-N+2}(M - N + 1; t) \\ D_2(M + 1; t) & D_1(M; t) & \cdots & D_{-N+3}(M - N + 2; t) \\ \vdots & \vdots & & \vdots \\ D_N(M + N - 1; t) & D_{N-1}(M + N - 2; t) & \cdots & D_1(M; t) \end{vmatrix}. \quad (34)$$

We can represent each element by a sum over the discrete time variable if $M \geqslant N$:

$$P(M, N, t) = \\ p^N \begin{vmatrix} \sum_{t'=M-1}^{t-1} D_0(M - 1; t') & \sum_{t'=M-2}^{t-1} D_{-1}(M - 2; t') & \cdots & \sum_{t'=M-N}^{t-1} D_{1-N}(M - N; t') \\ \sum_{t'=M}^{t-1} D_1(M; t') & \sum_{t'=M-1}^{t-1} D_0(M - 1; t') & \cdots & \sum_{t'=M-N+1}^{t-1} D_{2-N}(M - N + 1; t') \\ \vdots & \vdots & & \vdots \\ \sum_{t'=M+N-2}^{t-1} D_{N-1}(M + N - 2; t') & \sum_{t'=M+N-3}^{t-1} D_{N-2}(M + N - 3; t') \cdots & \sum_{t'=M-1}^{t-1} D_0(M - 1; t') \end{vmatrix}. \quad (35)$$

Instead of the partial integration we perform partial summation here and use the identity

$$\sum_{t'=t_1}^{t-1} b_{t'}(a_{t'+1} - a_{t'}) = -\sum_{t'=t_1}^{t-1} a_{t'+1}(b_{t'+1} - b_{t'}) + a_t b_t - a_{t_1} b_{t_1}. \quad (36)$$

for the second row with $a_t = t - 1$:

$$\sum_{t'=x}^{t-1} D_\nu(x, t') = -p \sum_{t'=x}^{t-1} t' D_{\nu-1}(x - 1, t') + (t - 1) D_\nu(x, t) - (x - 1) D_\nu(x, x)$$

$$= -p \sum_{t'=x-1}^{t-1} t' D_{\nu-1}(x - 1, t') + (t - 1) D_\nu(x, t). \quad (37)$$

For the third row we go further and apply (36) again with $a_t = t(t-1)/2$:

$$-p \sum_{t'=x-1}^{t-1} t' D_{\nu-1}(x-1,t') + (t-1) D_\nu(x,t)$$

$$= p^2 \sum_{t'=x-1}^{t-1} \frac{t'(t'+1)}{2} D_{\nu-2}(x-2,t') - p\frac{t(t-1)}{2} D_{\nu-1}(x-1,t)$$

$$+ p\frac{(x-1)(x-2)}{2} D_{\nu-1}(x-1.x-1) + (t-1) D_\nu(x,t)$$

$$= p^2 \sum_{t'=x-2}^{t-1} \frac{t'(t'+1)}{2} D_{\nu-2}(x-2,t') - p\frac{t(t-1)}{2} D_{\nu-1}(x-1,t)$$

$$+ (t-1) D_\nu(x,t). \tag{38}$$

In the fourth row we perform the partial summation once more with $a_t = (t-1)t(t+1)/3!$ and similarly for all the rows. Finally after adding suitable linear combination of the first $n-1$ rows to the n-th one we get

$$P(M,N,t) = \frac{p^{\frac{N(N+1)}{2}} (-1)^{\left[\frac{N}{2}\right]}}{0!1!\cdots(N-1)!}$$

$$\times \begin{vmatrix} \sum_{t'=M-1}^{t-1} D_0(M-1;t') & \sum_{t'=M-2}^{t-1} D_{-1}(M-2;t') & \cdots & \sum_{t'=M-N}^{t-1} D_{1-N}(M-N;t') \\ \sum_{t'=M-1}^{t-1} t' D_0(M-1;t') & \sum_{t'=M-2}^{t-1} t' D_{-1}(M-2;t') & \cdots & \sum_{t'=M-N}^{t-1} t' D_{1-N}(M-N;t') \\ \vdots & \vdots & & \vdots \\ \sum_{t'=M-1}^{t-1} t'^{N-1} D_0(M-1;t') & \sum_{t'=M-2}^{t-1} t'^{N-1} D_{-1}(M-2;t') & \cdots & \sum_{t'=M-N}^{t-1} t'^{N-1} D_{1-N}(M-N;t') \end{vmatrix}.$$

$$\tag{39}$$

We can set all the lower limits of the sums to 0 (see (B.8) and (B.10)) and write

$$P(M,N,t) = \frac{p^{\frac{N(N+1)}{2}} (-1)^{\left[\frac{N}{2}\right]}}{0!1!\cdots(N-1)!} \sum_{t_1,t_2,\cdots,t_N=0}^{t-1} t_1^0 t_2^1 t_3^2 \cdots t_N^{N-1}$$

$$\times \begin{vmatrix} D_0(M-1;t_1) & D_{-1}(M-2;t_1) & \cdots & D_{1-N}(M-N;t_1) \\ D_0(M-1;t_2) & D_{-1}(M-2;t_2) & \cdots & D_{1-N}(M-N;t_2) \\ \vdots & \vdots & & \vdots \\ D_0(M-1;t_N) & D_{-1}(M-2;t_N) & \cdots & D_{1-N}(M-N;t_N) \end{vmatrix}.$$

$$\tag{40}$$

Adding suitable linear combination of the first $n-1$ columns to the n-th one we get (by using (B.10))

$$
P(M, N, t) = \frac{p^{\frac{N(N+1)}{2}}(-1)^{\left[\frac{N}{2}\right]}}{0!1!\cdots(N-1)!} \sum_{t_1,t_2,\ldots,t_N=0}^{t-1} t_1^0 t_2^1 t_3^2 \ldots t_N^{N-1}
$$

$$
\times \begin{vmatrix}
D_0(M-1; t_1) & D_0(M-2; t_1) & \cdots & D_0(M-N; t_1) \\
D_0(M-1; t_2) & D_0(M-2; t_2) & \cdots & D_0(M-N; t_2) \\
\vdots & \vdots & & \vdots \\
D_0(M-1; t_N) & D_0(M-2; t_N) & \cdots & D_0(M-N; t_N)
\end{vmatrix}.
$$

(41)

Using the explicit form of D_0 (B.8) one arrives at

$$
P(M, N, t) = \frac{p^{MN}(1-p)^{-MN+\frac{N(N+1)}{2}}(-1)^{\left[\frac{N}{2}\right]}}{0!1!\cdots(N-1)!(M-1)!(M-2)!\cdots(M-N)!} \sum_{t_1,t_2,\ldots,t_N=0}^{t-1} (1-p)^{\sum_{j=1}^N t_j} t_1^0 t_2^1 t_3^2 \ldots t_N^{N-1}
$$

$$
\times \begin{vmatrix}
t_1(t_1-1)\cdots(t_1-M+2) & t_1(t_1-1)\cdots(t_1-M+3) & \cdots & t_1(t_1-1)\cdots(t_1-M+N+1) \\
t_2(t_2-1)\cdots(t_2-M+2) & t_2(t_2-1)\cdots(t_2-M+3) & \cdots & t_2(t_2-1)\cdots(t_2-M+N+1) \\
\vdots & \vdots & & \vdots \\
t_N(t_N-1)\cdots(t_N-M+2) & t_N(t_N-1)\cdots(t_N-M+3) & \cdots & t_N(t_N-1)\cdots(t_N-M+N+1)
\end{vmatrix}
$$

$$
= \frac{p^{MN}(1-p)^{-MN+\frac{N(N+1)}{2}}(-1)^{\left[\frac{N}{2}\right]}}{\prod_{j=1}^N (j-1)!(M-j)!} \sum_{t_1,t_2,\ldots,t_N=0}^{t-1} (1-p)^{\sum_{j=1}^N t_j} t_1^0 t_2^1 t_3^2 \ldots t_N^{N-1}
$$

$$
\times \prod_{j=1}^N t_j(t_j-1)(t_j-2)\cdots(t_j-M+N+1) \begin{vmatrix}
t_1^{N-1} & t_1^{N-2} & \cdots & 1 \\
t_2^{N-1} & t_2^{N-2} & \cdots & 1 \\
\vdots & \vdots & & \vdots \\
t_N^{N-1} & t_N^{N-2} & \cdots & 1
\end{vmatrix}
$$

$$
= \frac{p^{MN}(1-p)^{-MN+\frac{N(N+1)}{2}}}{\prod_{j=1}^N j!(M-j)!}
$$

$$
\times \sum_{t_1,t_2,\ldots,t_N=0}^{t-1} \prod_{j=1}^N t_j(t_j-1)(t_j-2)\cdots(t_j-M+N+1)(1-p)^{t_j} \begin{vmatrix}
t_1^{N-1} & t_1^{N-2} & \cdots & 1 \\
t_2^{N-1} & t_2^{N-2} & \cdots & 1 \\
\vdots & \vdots & & \vdots \\
t_N^{N-1} & t_N^{N-2} & \cdots & 1
\end{vmatrix}^2.
$$

(42)

Finally we get

$$P(M, N, t) = \frac{p^{MN}(1-p)^{-MN+\frac{N(N+1)}{2}}}{\prod_{j=1}^{N} j!(M-j)!}$$
$$\times \sum_{t_1, t_2, \cdots, t_N=0}^{t-1} \left(\prod_{j=1}^{N} \left(\prod_{k=0}^{M-N-1} (t_j - k) \right) (1-p)^{t_j} \right) \prod_{i<j} (t_i - t_j)^2.$$
$$\tag{43}$$

One can see that in the limit $t = \tilde{t}/p$, $p \to 0$, $\tilde{t} = \text{const.}$ (43) is equivalent to (22) as expected.

Expression (43) can be written as

$$P(M, N, t)$$
$$= Z(M, N)^{-1} \sum_{t_1, t_2, \cdots, t_N=0}^{t-1-M+N} \prod_{j=1}^{N} \left(\binom{t_j + M - N}{M - N} (1-p)^{t_j} \right) \prod_{i<j} (t_i - t_j)^2. \tag{44}$$

In the latter form $P(M, N, t)$ is similar to the same quantity of the usual discrete time ASEP (DTASEP) (see Proposition 1.3 of ref. 6), namely:

$$P(M, N, t) = P_{\text{DTASEP}}(M, N, t + N - 1). \tag{45}$$

This correspondence can be seen in a graphical representation of the dynamics (Fig. 1). The statistical weight coming from the path of the first particle is the same in both processes since its dynamics is identical in the two cases. The weight of the path of the Nth particle from 0 to t is the same as that of the normal DTASEP from 0 to $t + N - 1$ which explains the relation (45). However, this correspondence is valid only in the case of this special initial condition.

From the geometric interpretation it follows that (45) is valid for all values of M and N, although we derived this (on the level of formulas) only for $M \geqslant N$ (note that Proposition 1.3 of ref. 6 as well as (44) is valid only for $M \geqslant N$). The DTASEP has particle-hole symmetry which implies

$$P_{\text{DTASEP}}(M, N, t) = P_{\text{DTASEP}}(N, M, t), \tag{46}$$

from this we get

$$P(M, N, t) = P_{\text{DTASEP}}(M, N, t + N - 1) = P_{\text{DTASEP}}(N, M, t + N - 1)$$
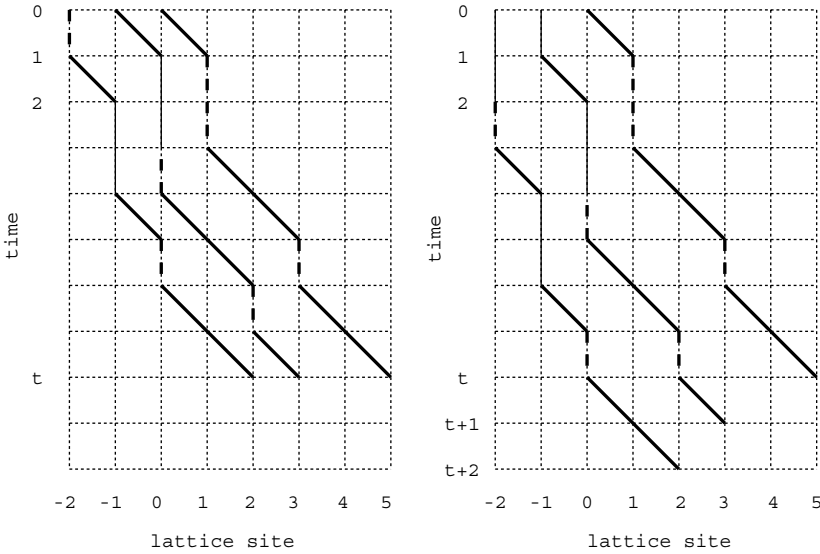$$= P(N, M, t + N - M). \tag{47}$$

Fig. 1. The first figure shows a possible history of three particles from time 0 to time t. We adjust probability p to the diagonal lines (thick solid lines) which correspond to particle hopping. The weight of the thick dashed vertical lines is $1 - p$ while the thin solid vertical lines have weight 1. The second figure shows the corresponding path configuration of the DTASEP. The path of the second (third ...) particle is shifted by one (two ...) time steps.

3.3. Asymptotic Form of the Distribution Function

Knowing the derivation of the asymptotic form of $P_{\text{DTASEP}}(M, N, t)$ (see Section 3 of ref. 6) it is rather easy to obtain similar results for this process. We repeat the result of Johansson for the discrete-time TASEP with parallel update with our notations:

$$\lim_{N \to \infty} P_{\text{DTASEP}}\left([\gamma N], N, N\omega_{\text{DTASEP}}(\gamma, p) + N^{1/3}\sigma(\gamma, p)s\right) = F_{\text{GUE}}(s) \tag{48}$$

with

$$\omega_{\text{DTASEP}}(\gamma, p) = \frac{\left(1 + \sqrt{(1-p)\gamma}\right)^2}{p} + \gamma, \tag{49}$$

$$\sigma(\gamma, p) = \frac{(1-p)^{1/6}\gamma^{-1/6}}{p}\left(\sqrt{\gamma} + \sqrt{1-p}\right)^{2/3}\left(1 + \sqrt{(1-p)\gamma}\right)^{2/3}. \tag{50}$$

This formula is derived for $\gamma \geqslant 1$ in ref. 6 but it is easy to show that it is valid also for $0 < \gamma < 1$. Using (46) one gets

$$\lim_{N \to \infty} P_{\text{DTASEP}} \left(\gamma N, N, N \omega_{\text{DTASEP}}(1/\gamma, p) \gamma + N^{1/3} \sigma(1/\gamma, p) \gamma^{1/3} s \right)$$
$$= F_{\text{GUE}}(s), \tag{51}$$

which is identical to (48) since $\omega_{\text{DTASEP}}(1/\gamma, p)\gamma = \omega_{\text{DTASEP}}(\gamma, p)$ and $\sigma(1/\gamma, p)\gamma^{1/3} = \sigma(\gamma, p)$.

To obtain the corresponding asymptotic result for the fragmentation process one goes through the same steps as ref. 6 (or simply use (45)) and (48). One finds

$$\lim_{N \to \infty} P \left([\gamma N], N, N \omega(\gamma, p) + N^{1/3} \sigma(\gamma, p) s \right) = F_{\text{GUE}}(s) \tag{52}$$

with

$$\omega(\gamma, p) = \frac{\left(1 + \sqrt{(1-p)\gamma}\right)^2}{p} + \gamma - 1 = \frac{\left(\sqrt{1-p} + \sqrt{\gamma}\right)^2}{p}. \tag{53}$$

This result is valid for any $\gamma > 0$. The corresponding identity is $\omega(1/\gamma, p)\gamma - 1 + \gamma = \omega(\gamma, p)$. We remark that the amplitude σ of the deviation is the same for both models. Using (5) this result translates into the expression for the distribution of the integrated current as

$$\lim_{t \to \infty} \text{Prob} \left[J(vt, t) > \eta(v, p)t - \kappa(v, p)t^{1/3}s \right]$$
$$= F_{\text{GUE}}(s) \quad \text{for } 0 < v < p \tag{54}$$

with

$$\eta(v, p) = \frac{1}{p} \left(\sqrt{1-v} - \sqrt{1-p} \right)^2 \tag{55}$$

and

$$\kappa(v, p) = \frac{(1-p)^{1/6}(1-v)^{1/3}(p-v)^{2/3} \left(p\sqrt{1-v} + v\sqrt{1-p} \right)^{2/3}}{\left(\sqrt{1-v} + \sqrt{1-p} \right)^{4/3} \left(p\sqrt{1-v} + v\sqrt{1-p} - v(1-p) \right)}. \tag{56}$$

Equivalently one obtains for the distribution of the height

$$\lim_{t \to \infty} \text{Prob} \left[h(vt, t) > (2\eta(v, p) + v)t - 2\kappa(v, p)t^{1/3}s \right]$$
$$= F_{\text{GUE}}(s) \quad \text{for} \ - p/(1 - p) < v < p. \tag{57}$$

4. CONCLUSIONS

We have shown that the distribution of the time-integrated current for the TASEP as well as for the totally asymmetric fragmentation process can be written as a determinant (34) obtained using Bethe ansatz. To show this one uses a determinant representation of the Bethe wave function which solves the master equation for a finite number of particles. After appropriate scaling the current distribution is given by the distribution of the largest eigenvalue of a random matrix ensemble. This observation may lead to better understanding of the relation between the random matrix theory and the Bethe ansatz and suggests that also other integrable hopping processes may be treated in a similar fashion. The main task that remains is the derivation of suitable determinant representations for conditional probabilities for such processes. In the scaling limit of the fragmentation process the distribution of the time-integrated current around its mean converges to the same Tracy–Widom distribution for the Gaussian unitary ensemble found previously for the TASEP, thus confirming universality of this quantity.

APPENDIX A: *F FUNCTIONS*

Definition:

$$F_p(n; t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-(1 - e^{-ik})t} \left(1 - e^{i(k+i0)} \right)^{-p} e^{ikn} \, dk, \tag{A.1}$$

$$= \frac{1}{2\pi i} \oint_{|z|=1-0} e^{-(1 - z^{-1})t} (1 - z)^{-p} z^{n-1} \, dz. \tag{A.2}$$

Differentiation, integration:

$$\frac{d}{dt} F_p(n; t) = F_{p-1}(n - 1; t), \tag{A.3}$$

$$\int_{t_1}^{t_2} F_p(n; t) = F_{p+1}(n + 1, t_2) - F_{p+1}(n + 1, t_1). \tag{A.4}$$

Summation:

$$\sum_{n=n_1}^{n_2} F_p(n;t) = F_{p+1}(n_1;t) - F_{p+1}(n_2+1;t), \qquad (A.5)$$

$$F_p(n;t) = F_{p+1}(n;t) - F_{p+1}(n+1;t), \qquad (A.6)$$

$$\sum_{n=n_1}^{\infty} F_p(n;t) = F_{p+1}(n_1;t). \qquad (A.7)$$

For $p \leqslant 0$ and $n \geqslant 0$:

$$F_p(n;t) = e^{-t} \sum_{m=0}^{-p} (-1)^m \binom{-p}{m} \frac{t^{m+n}}{(m+n)!}. \qquad (A.8)$$

$F_p(n;t)$ can be written as a (finite or infinite) sum also in other regions of the (n, p) parameter space (see ref. 17) but those formulas are not used here.

For $n \geqslant 0$ (and any p):

$$F_p(n;0) = \delta_{n,0}. \qquad (A.9)$$

APPENDIX B: *D FUNCTIONS*

Definition:

$$D_q(n,t) = \frac{1}{2\pi} \int_0^{2\pi} dk \left(1 - p + pe^{-ik}\right)^t \left(1 - e^{i(k+i0)}\right)^{-p} e^{ikn}, \qquad (B.1)$$

$$= \frac{1}{2\pi i} \oint_{|z|=1-0} dz \left(1 - p + \frac{p}{z}\right)^t (1-z)^{-q} z^{n-1}. \qquad (B.2)$$

Discrete time derivative and summation for t:

$$D_q(n,t+1) - D_q(n,t) = p D_{q-1}(n-1,t), \qquad (B.3)$$

$$\sum_{t=t_1}^{t_2} D_q(n,t) = \frac{1}{p} \left(D_{q+1}(n+1,t_2+1) - D_{q+1}(n+1,t_1)\right). \qquad (B.4)$$

Summation for n:

$$\sum_{n=n_1}^{n_2} D_q(n;t) = D_{q+1}(n_1;t) - D_{q+1}(n_2+1;t), \tag{B.5}$$

$$D_q(n;t) = D_{q+1}(n;t) - D_{q+1}(n+1;t), \tag{B.6}$$

$$\sum_{n=n_1}^{\infty} D_q(n;t) = D_{q+1}(n_1;t). \tag{B.7}$$

Explicit form of $D_q(n,t)$ for $n,t \geqslant 0$:

$$D_0(n,t) = \begin{cases} 0, & t < n, \\ \binom{t}{n}(1-p)^{t-n}p^n, & 0 \leqslant n \leqslant t, \\ 0 & n \leqslant 0, \end{cases} \tag{B.8}$$

$$D_{q>0}(n,t) = \sum_{j=0}^{\infty} \binom{q+j-1}{j} D_0(n+j,t), \tag{B.9}$$

$$D_{q\leqslant 0}(n,t) = \sum_{j=0}^{-q} \binom{-q}{j}(-1)^j D_0(n+j,t), \tag{B.10}$$

$$D_q(n,n) = D_0(n,n) = p^n. \tag{B.11}$$

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Note added: After completion of this work we learned that the result of Section 2.2. has been found also by Nagao and Sasamoto.[23]

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